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Integrated Structural Design and Vibration Suppression Using Independent Modal Space Control

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The integrated design of a structure and its control system is treated as a multiobjective optimization problem. Structural mass and a quadratic performance index constitute the vector objective function. The closed-loop performance index is taken as the time integral of the Hamiltonian. Constraints on natural frequencies, closed-loop damping, and actuator forces are also considered. Derivatives of the objective and constraint functions with respect to structural and control design variables are derived for a finite element beam model of the structure and constant feedback gains determined by independent modal space control. Pareto optimal designs generated for a simple beam demonstrate the benefit of solving the integrated structural and control optimization problem.

Introduction

THE fields of structural optimization and optimal control have made significant strides with the advent of the computer age. Each discipline has independently generated useful methods and computational tools for design. Yet, they share an approach in which certain system parameters or design variables are selected so as to optimize an objective function subject to a given set of constraints. In structural optimization, the objective function is often related to the cost of the structure, and it may involve the weight of the structure or the volume of material. The design variables typically characterize the material distribution or geometry of the structure. Any quantity characterizing the response of the structure, such as stress, displacement, or frequency, may be constrained to preclude structural failure. The structural response may be static or dynamic, although time usually plays no particular role in the formulation of the structural optimization problem.[1,2] In contrast, in modern optimal control a control law optimizing some performance index (the objective function) over a given time interval is synthesized. In the linear quadratic regulator (LQR) theory,[3] control gains relating actuator forces to sensor outputs by means of a linear transformation are typical control design variables. The performance index is rendered independent of time by integrating over the control time.

In common design practice, structural design precedes control design. The mathematical model for the structural optimum, or at least the final structural design, constitutes the plant for the control design. The control designer then proceeds to synthesize the optimal control system for the given plant. Uncertainties in structural modeling of the plant are often considered to ensure that the control system is robust with respect to plant errors. The possibility exists that the plant can be altered so as to enhance performance in conjunction with both structural and control optimization. As a result of difficulties in lifting and deploying heavy objects such as space stations, which can contain large solar arrays, antennas and precision laser, or optical systems, the spacecraft structures must be highly flexible. Moreover, stringent performance requirements for pointing accuracy, vibration suppression, shape control, etc., demand active controls to augment any passive damping. The goal of integrated design is to take advantage of any synergistic interaction between the flexible structure and its active control system.

The integrated structural and control design problem was first considered by Hale et al.[4] Messac and Turner[5] emphasized transformation of the state space to modal coordinates and the dependence of the weighting matrices on the modes. In formulating the combined optimization problem Salama et al.[6] established a precedent followed by many others. They eliminated control design variables by considering steady-state (constant) gains and selecting particular weighting matrices (identity) for the quadratic performance index.

Haftka et al.[7] minimized control effort through structural changes while maintaining specified damping ratios. However, the damping ratios of the lower modes decreased, and the frequency response magnitude and decay time increased. In another early study, Venkayya and Tischler[8] reached a different conclusion, namely, that a nominal truss model and a structurally optimal truss had nearly the same control design for regulating a disturbance. They considered a disturbance dominated by the first mode, but both structural designs had the same fundamental eigenvalue, perhaps explaining the similar control designs. Miller and Shim[9] also considered the same class of disturbances, i.e., an initial displacement representing the response to a static load.

Rew and Junkins[10] searched for optimal state and control weighting matrices for the quadratic performance index but did not consider any structural variables. Bodden and Junkins[11] minimized some unspecified robustness measure while placing closed-loop eigenvalues in a desired region. When the structural variables were included, a sequential approach was used. McLaren and Slater[12] employed a similar homotopy continuation method that converged only after about 750 complete analyses for one case, and after 10 h of CPU time on a Cray Y-MP supercomputer for another.

Lust and Schmit[13] searched for structural variables and linear output feedback gains on equal footing for the steady-state response to deterministic harmonic loading. Special attention was paid to developing explicit approximations to the highly nonlinear implicit response functions. Later, Thomas and Schmit[14] extended this formulation to problems with noncollocated actuators and sensors, as well as stability constraints on the complex eigenvalues. Sepulveda et al.[15] extended to complex eigenvalues Canfield's Rayleigh quotient approximation (RQA) for eigenvalues of conservative systems.[16] Based on the complex RQA, Thomas et al.[17] improved the approximations for damping and control forces.

Only Belvin and Park[18] appear to have taken advantage of inde-

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pendent modal space control[3] (IMSC) to treat the integrated structural/control problem. The authors did not claim to have found the integrated optimum, rather they used IMSC to decouple structural and control design by determining the primary influence of the structural design on the control objective.

A novel approach proposed by Messac et al.[19] combined disturbance rejection and command following performance errors in a quadratic performance norm. One of the final designs was characterized as nonintuitive, because the second frequency was reduced by half, whereas in separable design structural optimization would normally stiffen the second mode, increasing its frequency.[18] A comparison of designs obtained for two simple truss structures was made by Rao[20] for various objective functions.

Problem Statement

We are concerned with a linear, time-invariant, distributed-parameter system. Following spatial discretization, the equations of motion are given by

$$M(v_s)\ddot{q}(t) + C(v_s)\dot{q}(t) + K(v_s)q(t) = Du(v_s, v_c, t) \tag{1}$$

where q is the configuration vector, M the mass matrix, C the damping matrix, and K the stiffness matrix, all parameterized by the n_s vector v_s of structural variables. The system is regulated by a vector of control forces $u(v_s, v_c, t)$ which is also parameterized by the n_c-vector v_c of control design variables, and transmitted to the structure according to the load distribution matrix D. The structural, and control vectors can be combined into the complete design vector $v = [v_s^T \ v_c^T]^T$.

Modal Space Control

If the structure is subject to proportional viscous damping, the uncontrolled system can be decoupled by means of the classical modal transformation. By the expansion theorem,[21] this transformation is given by

$$q(t) = \sum_{i=1}^{n_m} \phi_i \eta_i(t) = \Phi \eta \tag{2}$$

where Φ is the classical modal matrix and η the vector of modal coordinates. The eigenvectors are orthogonal with respect to the mass matrix and can be normalized so as to satisfy $\Phi^T M \Phi = I$, $\Phi^T K \Phi = \Omega^2$, where Ω^2 is the diagonal matrix of eigenvalues. Synthesizing a control law is more convenient when the left side of Eq. (1) is transformed to diagonal form. To this end we insert Eq. (2) into Eq. (1), premultiply by Φ^T, assume that damping is of the proportional type,[21] and obtain

$$\ddot{\eta}(t) + 2\zeta\Omega(v_s)\dot{\eta}(t) + \Omega^2(v_s)\eta(t) = \Phi^T(v_s)Du(v_s, v_c, t) = f(t) \tag{3}$$

where ζ is the diagonal matrix of modal damping factors, assumed to be constant for the class of structures under consideration, and $f(t)$ is the modal force vector.

Typically, not all n_m modes need be controlled, so the control design problem can be truncated. To this end, we partition the decoupled equations into n_C equations for the controlled modes and n_R equations for the residual modes. Consistent with this, the modal matrix, the modal coordinate vector, and the modal force vector have the partitioned forms $\Phi = [\Phi_C \ \Phi_R]$, $\eta = [\eta_C^T \ \eta_R^T]^T$, and $f(t) = [f_C(t)^T \ f_R(t)^T]^T$, respectively. The controlled equations in Eq. (3) can be transformed to state space. To this end, we introduce the state vector $x = [\eta_C^T \ \dot{\eta}_C^T]^T$. Then, the state equations for the controlled modes can be written in the matrix form

$$\dot{x} = Ax + Bu \tag{4}$$

where

$$A = \begin{bmatrix} 0 & I \\ -\Omega_C^2 & -2\zeta_C\Omega_C \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \Phi_C^T D \end{bmatrix} \tag{5}$$

are coefficient matrices. For linear feedback of the controlled modes, the control law is

$$f_C(t) = \Phi_C^T Du(t) = -G\eta_C - H\dot{\eta}_C \tag{6}$$

where the modal control gain matrices G and H are constant in the steady-state case. To implement modal feedback control the modal displacements and velocities must be extracted from the system outputs. Application of the second part of the expansion theorem[21] provides the transformation of physical to modal coordinates

$$\eta_C(t) = \Phi_C^T M q(t) \tag{7a}$$

$$\dot{\eta}_C(t) = \Phi_C^T M \dot{q}(t) \tag{7b}$$

in which the $q(t)$ are deterministic physical coordinates. State estimation is not considered here; however, if at least as many sensed outputs are available as controlled modes, modal state estimation is not an obstacle.[3] Meirovitch et al.[22] successfully demonstrated the use of modal filters based on Eqs. (7) for control of an experimental beam by means of IMSC.

Although they are not controlled, the residual modes receive the excitation

$$f_R(t) = \Phi_R^T Du(t) \tag{8}$$

The closed-loop system is described by

$$\dot{x} = A_c x \tag{9}$$

where

$$A_c = \begin{bmatrix} 0 & I \\ -\bar{G} & -\bar{H} \end{bmatrix} \tag{10}$$

is the closed-loop plant matrix, in which

$$\bar{G} = \Omega_C^2 + G, \qquad \bar{H} = 2\zeta_C\Omega_C + H \tag{11}$$

Independent Modal Space Control

The general idea of the IMSC method is that the control force for a given mode depends only on the modal displacement and velocity of that mode. Hence, the independence of open-loop modal equations is preserved for the closed-loop equations. For self-adjoint systems, the $2n_C \times 2n_C$ matrix Riccati equation for linear optimal control reduces to a series of n_C independent 2×2 matrix equations for each of the controlled modes. It follows that, in the case of IMSC, the modal gain matrices G and H in Eq. (6) are diagonal. Equation (6) yields the modal control forces, from which we obtain the actuator forces

$$u(t) = (\Phi_C^T D)^{-1} f_C(t) \tag{12}$$

When the number of controlled modes and actuators is not the same, the exact inverse $(\Phi_C^T D)^{-1}$ in Eq. (12) can be replaced by a pseudoinverse, but the modal forces will be only approximately independent, depending how close $\Phi_C^T D$ is to a square matrix.

In optimal IMSC, the weighting matrices for the modal space quadratic performance index are assumed to be diagonal, so that the performance index for the steady-state case can be expressed as the sum of independent modal performance indices in the form

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + f_C^T R f_C)\, dt$$

$$= \frac{1}{2}\sum_{r=1}^{n_C}\int_0^\infty (\dot{\eta}_r^2 + \omega_r^2\eta_r^2 + r_r f_r^2)\, dt \tag{13}$$

As a result, the $2n_C \times 2n_C$ matrix algebraic Riccati equation reduces to n_C independent 2×2 matrix algebraic Riccati equations. The latter have the analytic solution

$$g_r = \omega_r\sqrt{\omega_r^2 + r_r^{-1}} - \omega_r^2$$
$$h_r = \sqrt{r_r^{-1} + 2g_r}, \qquad r = 1, 2, \ldots, n_C \tag{14}$$

One choice of control design variables is the modal control force weighting factors $v_c = [r_1 r_2 \ldots r_{n_C}]^T$, because they determine the modal gains for IMSC optimal control uniquely. Alternatively, all of the r_r could be eliminated by using Eqs. (14) to solve for each h_r in terms of g_r, or vice versa. In the former case,

$$h_r(g_r, \omega_r) = \sqrt{r_r^{-1} + 2g_r}, \qquad r_r^{-1} = \left(\frac{g_r + \omega_r^2}{\omega_r}\right)^2 - \omega_r^2 \tag{15}$$

whereas in the latter case,

$$g_r(h_r, \omega_r) = \left[\sqrt{4 + \left(\frac{h_r}{\omega_r}\right)^2} - 2\right]\omega_r^2 \tag{16}$$

Now, either the g_r or the h_r can be taken as control design variables.

Pareto Optimal Solutions

A natural approach to integrating design requirements from two or more disciplines is to optimize a vector of cost functions. Ideally, all of the criteria in the vector would be optimized simultaneously. Because all criteria cannot be optimized at once, an understanding of what constitutes a vector optimum is required. Following Koski,[23] we state the multiobjective (multicriteria or vector) optimization problem as

$$F^*(v^*) = \min_{v \in \Omega} F(v) \tag{17a}$$

where $F:\Omega \to R^m$ is a vector objective function

$$F(v) = \lfloor F_1(v) \; F_2(v) \; \ldots \; F_m(v) \rfloor^T \tag{17b}$$

and its components $F_i:\Omega \to R$, $i = 1, 2, \ldots, m$ are the criteria. The design variable vector v belongs to the feasible set $\Omega \subset R^n$, defined by the vectors h and g of equality and inequality constraints, respectively, as

$$\Omega = \{v \in R^n | h(v) = 0, g(v) \le 0\} \tag{17c}$$

where vector inequalities are understood to apply individually to each component of the vector. (Distinction between the vectors g and h for constraint functions and modal gains g_r and h_r should be clear from the context of the equations.) In general, no unique point exists that optimizes all m criteria simultaneously. Pareto defined the vector optimum v^* as that for which there exists no feasible vector v that would decrease some criterion without simultaneously increasing at least one other criterion.

A popular technique for generating Pareto minima is the constraint method.[23] In this method the vector optimization is replaced by

$$F_k^*(v^*) = \min_{v \in \Omega \cap \Omega_k(\varepsilon)} F_k(v) \tag{18a}$$

where

$$\Omega_k(\varepsilon) = \{v : F_i(v) \le \varepsilon_i, \; i \ne k\} \tag{18b}$$

and

$$\varepsilon \in E_k = \{\lfloor \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{k-1} \varepsilon_{k+1} \ldots \varepsilon_m \rfloor^T\} : \Omega(\varepsilon) \ne \emptyset \tag{18c}$$

One criterion is taken as a scalar objective function whereas the remaining are constrained by a set of constants ε_i such that a feasible solution exists. Again, parametric variation of the ε_i generates a set of Pareto minima. An important implication for the problem under consideration is that any single objective function may be minimized whereas the others are treated as constraints. The choice of criterion is at our discretion. The same Pareto optimal designs should be generated by minimizing either structural weight or the quadratic performance index, for example, while constraining the other. Of course, this assumes that each scalar optimization finds the global minimum or that the same set of local minima will be found for each choice of objective function. The constraint method is the basis for seeking Pareto optimal solutions of this research. Thus, the focus is on determining efficient methods to solve Eqs. (18) for the integrated structural and control problem.

Theoretical Development

Performance Index for Integrated Structural and Control Design

In using the LQR theory[3] for optimal control, one determines a control law that minimizes a quadratic performance index for a given positive semidefinite state weighting matrix Q and a positive definite control weighting matrix R. The choice of weighting matrices is otherwise at the discretion of the control designer. Each choice produces a different optimal control. In practice, the designer alters Q and R to balance system performance and control effort. As an example, one particular choice might result in saturation of an actuator in some simulation. Then, the designer might increase the weighting factor(s) associated with that actuator. In this sense, the performance index is not used to compare candidate control laws. Instead, other criteria are used as a basis for comparison in the simulations. The dilemma is compounded in integrated structural and control design, in which performance for different plants and control laws must be compared. Therefore, the question of an appropriate performance measure for controls is a critical one. We propose here that the time integral of the total system energy be used as the single, unique, and physically meaningful performance measure. The total system energy consists of the kinetic and elastic potential energy and the energy expended by the control system. As mentioned in the literature review, the structural energy has been used frequently to define a unique state weighting matrix Q. Defining the control weighting matrix R is more arbitrary. Two interpretations of "control effort" are considered next.

The control effort in the performance index is taken to represent the energy expended by the control system, which is often assumed to be proportional to the square of the control inputs. When the input represents an actuator force, the energy to actuate certain mechanical systems might be proportional to the square of each actuator's force. In this case, we expect the energy expended by each actuator to be independent of the others, in which case R is diagonal. Assuming that the proportionality factors of each actuator are known a priori, we consider a performance measure for the system defined by Eq. (1) in the form

$$J = \frac{1}{2}\int_0^\infty \left(\begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}^T \begin{bmatrix} K_s & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + u^T R u \right) dt \tag{19}$$

where R is a unique weighting matrix of factors for the control energy.

Before proceeding any further, let us consider alternative measures for the control effort. The weighting matrix $R = D^T K^{-1} D$ was proposed by Venkayya and Tischler[8] and employed by Rao[20] and Belvin and Park.[18] This weighting matrix provides a quasistatic measure of the energy imparted to the structure by the control system. It can be derived by assuming that the forces are applied quasistatically to the structure by the control $Du(t)$. The elastic energy imparted to the structure has the expression

$$(1/2)[Du(t)]^T \tilde{q}(t) = (1/2)[Du(t)]^T K^{-1} Du(t)$$
$$= (1/2)\, u^T(t) R u(t) \tag{20}$$

where $\tilde{q}(t)$ represents the static displacement due to the quasi-static force $\boldsymbol{Du}(t)$. However, one could as well consider the energy imparted to the structure by the control force vector $\boldsymbol{Du}(t)$ applied impulsively. Both are approximate measures of the actual energy used by the control system to do work on the structure.

Next, we consider the actual work done by the actuator forces, or

$$W = \int_0^t \dot{\boldsymbol{q}}^T F(t)\,\mathrm{d}t = \int_0^t \dot{\eta}^T f(t)\,\mathrm{d}t = W_C + W_R \tag{21}$$

where

$$W_C = \int_0^t \dot{\eta}_C f_C(\tau)\,\mathrm{d}t, \qquad W_R = \int_0^t \dot{\eta}_R f_R(\tau)\,\mathrm{d}t \tag{22}$$

are the work done by the controlled modes and by the residual modes, respectively. At this point, we define the useful work as the work done by the actuator forces on the controlled modes, i.e., W_C, and note that for IMSC we can evaluate W_C analytically. Because the modal forces are linear in the state variables, we assume that the integral has a quadratic form, or

$$\int_0^t \dot{\eta}_C^T(\tau)\, f_C(\tau)\,\mathrm{d}\tau = \boldsymbol{x}^T(t) R \boldsymbol{x}(t) \tag{23}$$

in which $\boldsymbol{x} = [\eta_C^T \ \dot{\eta}_C^T]^T$ is the modal state vector. Differentiating both sides of Eq. (23) with respect to time, we have

$$\dot{\eta}_C^T(t)\, f_C(t) = \dot{x}^T(t) R x(t) + x^T(t) R \dot{x}(t) \tag{24}$$

Substituting $x = [\eta_C^T \ \dot{\eta}_C^T]^T$ and Eq. (6) into the left side of Eq. (24) and Eq. (9) into the right side, we obtain

$$\boldsymbol{x}^T \begin{bmatrix} 0 & -(1/2)G \\ -(1/2)G & -H \end{bmatrix} \boldsymbol{x} = \boldsymbol{x}^T [A_c^T R + R A_c] \boldsymbol{x} \tag{25}$$

Equation (25) will be satisfied for an arbitrary state only if the matrices in brackets are equal. Because for IMSC the various submatrices are diagonal, insertion of Eq. (10) into Eq. (25) yields

$$R = \frac{1}{2} \begin{bmatrix} \bar{G}\bar{H}^{-1} H - G & 0 \\ 0 & \bar{H}^{-1} H \end{bmatrix} \tag{26}$$

where $\bar{G}$ and $\bar{H}$ are defined by Eqs. (11). When no natural damping is present $\bar{H} = H$ and Eq. (26) reduces to

$$R = \frac{1}{2} \begin{bmatrix} \Omega_C^2 & 0 \\ 0 & I \end{bmatrix} \tag{27}$$

which is precisely equal to the modal state weighting matrix Q rendering $(1/2)\boldsymbol{x}^T Q \boldsymbol{x}$ the sum of kinetic and strain energy. This result is to be expected, because for a conservative structural system the change in energy from the initial time to a quiescent state at the final time must be dissipated entirely by the control system. Hence, if we define control effort as the useful work done by the control system, it is sufficient to merely include the state weighting matrix in the performance index. It is important to stress here that Eq. (19) with $R = 0$ represents a performance index for comparing two systems and not a performance index used to derive LQR control gains.

In summary, in seeking an appropriate and physically meaningful control performance measure we arrived at two alternatives. One employs the unique matrix R in Eq. (19) yielding the energy expended by the actuators over time, assuming that such a characterization of the control system is available. In the absence of knowledge about R, the second alternative is to consider the useful work done by the control system over a given time interval. For a conservative plant the useful work is simply the change in internal energy of the plant. Therefore, when comparing two systems, it is sufficient to use the unique state weighting matrix that produces the total structural energy and let $R = 0$ when comparing two systems. Of course, the second choice ignores control spillover, which must be small for any feasible system. We also assumed the control gains were such that the actuators did not reach saturation. Next, we derive the time-invariant form of the performance index and formulate constraints for the actuator forces.

Independent Model Space Control Time-Invariant System Energy

We wish to minimize the time integral of the system energy, Eq. (19), as a function of the vector of structural parameters $\boldsymbol{v}_s$ determining the natural frequencies and the vector of control parameters $\boldsymbol{v}_c$ determining the control gains. Because the plant and control are taken to be time invariant, the performance index can be expressed without regard to time, assuming that the closed-loop system is stable. The performance index, Eq. (19), can be written in the partitioned modal state space form

$$J = \begin{Bmatrix} \eta_C(0^+) \\ \dot{\eta}_C(0^+) \end{Bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{Bmatrix} \eta_C(0^+) \\ \eta_C(0^+) \end{Bmatrix} \tag{28}$$

The positive definite symmetric coefficient matrix P is a solution to the Lyapunov equation

$$Q_c + A_c^T P + P A_c = 0 \tag{29}$$

for

$$Q_c = \begin{bmatrix} \Omega^2 & 0 \\ 0 & I \end{bmatrix} + [G_\eta \ \ H_\eta]^T R\, [G_\eta \ \ H_\eta] \tag{30}$$

where G_η and H_η relate the actuator forces to modal coordinates according to

$$u(t) = -G_\eta \eta_C - H_\eta \dot{\eta}_C \tag{31}$$

The use of IMSC in formulating Eqs. (28–30) has significant implications. First, the actuator gains can be synthesized from the modal control gains by means of Eq. (6) or

$$G_\eta = (\Phi_C^T D)^{-1} G, \qquad H_\eta = (\Phi_C^T D)^{-1} H \tag{32}$$

As a result, even if R is diagonal in Eq. (30), the modes recouple Eq. (29) via Eqs. (32). However, choosing $R = 0$ becomes quite attractive when using IMSC. Then Q_c becomes diagonal, thus decoupling Eq. (29) and permitting an analytical determination of the diagonal partition matrices of P. Indeed, inserting

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{33}$$

and Eqs. (10), (11), and (30) in to Eq. (29), we obtain

$$p_{11_i} = \frac{\omega_i^2 (2\zeta_i \omega_i + h_i)}{2(\omega_i^2 + g_i)} + \frac{2\omega_i^2 + g_i}{2(2\zeta_i \omega_i + h_i)} \tag{34a}$$

$$p_{12_i} = \frac{\omega_i^2}{2(\omega_i^2 + g_i)} \tag{34b}$$

$$p_{22_i} = \frac{2\omega_i^2 + g_i}{2(\omega_i^2 + g_i)(2\zeta_i \omega_i + h_i)}, \qquad i = 1, 2, \ldots, n_C \tag{34c}$$

A critical assumption must be examined for properly minimizing the integrated objective function, Eq. (28). An arbitrary external disturbance for a fixed plant is often assumed to induce an arbitrary initial state of unknown direction but constant magnitude. Minimizing Eq. (28) for this case is then shown to be equivalent to

$$\min J \Leftrightarrow \min\,[\mathrm{tr}(P)] \tag{35}$$

In earlier approaches to the integrated problem, this assumption has often been made in a casual manner. However, if design of the plant itself is under consideration, the initial state depends on the structural properties. Indeed, an arbitrary external disturbance of given magnitude produces a corresponding initial state depending on the as yet undetermined stiffness and mass of the structure. Therefore, the sensitivity of the initial state $\partial x_0/\partial v_{si}$ cannot be ignored. Calculating $\partial x_0/\partial v_{si}$ involves the assumption of a certain disturbance, e.g., an impulsive force of given magnitude or at least assumed statistics governing the nature of the disturbance. The same assumptions apply to calculating actuator forces, as well. For a given disturbance, the derivative of the performance index, Eq. (28), with respect to structural and control variables is

$$\frac{\partial J}{\partial v_i} = x_0^T\frac{\partial P}{\partial v_i}x_0 + 2x_0^T P\frac{\partial x_0}{\partial v_i} \tag{36}$$

Sensitivity of the first term requires differentiation of Eq. (29). The second term in Eq. (36) involves structural variables only. Both terms are included in the following derivation for the IMSC performance index.

Consider an impulsive disturbance represented by the force vector $\hat{\boldsymbol{Q}}\delta(t)$ where $\hat{\boldsymbol{Q}}$ is the intensity vector and $\delta(t)$ is the Dirac delta function. Solution of the modal counterpart to Eq. (1) with the right side replaced by the impulsive force $\hat{\boldsymbol{Q}}\delta(t)$ results in the initial modal velocities

$$\dot{\eta}(0^+) = \Phi^T(v_s)\hat{\boldsymbol{Q}} \tag{37}$$

By substitution of Eq. (37) into Eq. (28), the performance index can be simplified to

$$J = \dot{\eta}^T(0^+)P_{22}\dot{\eta}^T(0^+) = (\Phi_C^T\hat{\boldsymbol{Q}})^T P_{22}(\Phi_C^T\hat{\boldsymbol{Q}})$$

$$= \hat{\Phi}^T P_{22}\hat{\Phi} = \sum_{i=1}^{n_C} p_{22_i}\hat{\phi}_i^2 \tag{38}$$

where $\hat{\phi}_i$ is the ith component of the vector $\hat{\Phi} \equiv \Phi_C^T\hat{\boldsymbol{Q}}$, so that the control objective represented by Eq. (38) a simple weighted sum of the diagonal elements of $P_{22}(\boldsymbol{v}_s, \boldsymbol{v}_c)$. It should be pointed out here that an initial condition due to an impulse is more realistic than a static preload and applies to unconstrained as well as to constrained structures. Sensitivity of the integrated IMSC performance index for an impulsive disturbance, Eq. (38), is considered in the next section.

Independent Modal Space Control Performance Index Derivatives

Differentiation of Eq. (38) with respect to the structural variables leads to

$$\frac{\partial J}{\partial v_{sk}} = \sum_{i=1}^{n_C}\left(2p_{22_i}\hat{\phi}_i\frac{\partial\hat{\phi}_i}{\partial v_{sk}} + \frac{\partial p_{22_i}}{\partial v_{sk}}\hat{\phi}_i^2\right) \tag{39}$$

The partial derivative term $\partial\hat{\phi}_i/\partial v_{sk}$ in Eq. (39) requires natural eigenvector derivatives. An explicit expression for the partial derivative term

$$\frac{\partial p_{22_i}}{\partial v_{sk}} = \frac{h_i}{4(\omega_i^2+g_i)^2\sqrt{4+h_i^2/\omega_i^2}}\frac{\partial\omega_i^2}{\partial v_{sk}} \tag{40}$$

was derived by differentiating Eq. (34c). The structural frequency derivative in Eq. (40) is well known.[1] Note that its coefficient on the right side of Eq. (40) is positive,

$$\frac{\partial p_{22_i}}{\partial\omega_i^2} > 0 \tag{41}$$

so that, ignoring the modal sensitivity appearing in Eq. (39), we conclude that a reduction of the performance index implies a reduction of structural frequencies, i.e., a softening of the structure. This result is in opposition to the heuristic conclusion reached by Belvin and Park,[18] but agrees with the observation of Messac et al.[19]

Differentiation of Eq. (38) with respect to the control design variables leads to

$$\frac{\partial J}{\partial v_{ck}} = \frac{\partial p_{22_k}}{\partial v_{ck}}\hat{\phi}_k^2 \tag{42}$$

The partial derivative with respect to a control design variable is the partial derivative with respect to the IMSC modal rate gain, $v_{ck} = h_k$. Hence, by differentiating Eq. (34c) and using Eq. (16) for optimal IMSC gains, the partial derivative on the right side of Eq. (42) can be shown to have the form

$$\frac{\partial p_{22_i}}{\partial h_i} = \frac{-\omega_i^2}{2(\omega_i^2+g_i)^2\sqrt{4+h_i^2/\omega_i^2}} - \frac{\omega_i^2+g_i/2}{h_i^2(\omega_i^2+g_i)} \tag{43}$$

and note that $\partial p_{22_i}/\partial h_i < 0$ so that decreasing the performance index implies increasing the modal gains, as expected. However, the modal gains will be prevented from becoming infinitely large by constraints on the actuator forces posed in the next section.

Actuator Force Constraints

Actuator forces are constrained to be smaller than some maximum magnitude for all time. To create a finite number of constraints for the integrated optimization, the actuator inputs (forces) can be evaluated at only a few peak times. Peak actuator forces occur either at $t = 0$ or when the derivative of the force with respect to time vanishes. In the first case, the peak inputs for each actuator are easily determined from $\boldsymbol{u}(0^+) = -\hat{G}x_0$. Inserting Eqs. (6) and (37) into Eq. (12), the initial IMSC actuator forces corresponding to an initial velocity caused by an impulse of intensity $\hat{\boldsymbol{Q}}$ are

$$\boldsymbol{u}(0^+) = -(\Phi_C^T D)^{-1} H\Phi_C^T\hat{\boldsymbol{Q}} \tag{44}$$

Subsequent peak inputs first require determination of the peak times from the transcendental equation resulting from differentiating the scalar form of Eq. (12)

$$0 = \left.\frac{\partial u_j(t)}{\partial t}\right|_{\hat{t}} = -\hat{G}_j A_c\boldsymbol{x}(\hat{t}), \qquad \forall \in N_u \tag{45}$$

where $\hat{G}_j$ represents the jth row of the gain matrix $\hat{G}$. Solution of Eq. (45) generates a set of p_j peak times $T_j = \{\hat{t}_1, \ldots, \hat{t}_{pj}\}$ for each of the n_u inputs. For IMSC the modal states are known explicitly in terms of the design variables and time.[24]

Actuator Force Derivatives

Consider first the actuators forces at time $t = 0^+$ due to an impulse. The initial state vector

$$\boldsymbol{x}(0^+) = \begin{Bmatrix} 0 \\ \Phi_C^T\hat{\boldsymbol{Q}} \end{Bmatrix} \tag{46}$$

does not vary with time or control variables, only with structural variables. Therefore, we divide the possible derivatives into those that depend on structural variables $\boldsymbol{v}_s$ and those that depend on control variables $\boldsymbol{v}_c$. Differentiating Eq. (44) with respect to structural variables first and noting that H does not depend explicitly on v_{sk}, we have

$$\frac{\partial \boldsymbol{u}(0^+)}{\partial v_{sk}} = -(\Phi_C^T D)^{-1}\left[H\frac{\partial(\Phi_C^T\hat{\boldsymbol{Q}})}{\partial v_{sk}} - \frac{\partial\Phi_C^T}{\partial v_{sk}}D(\Phi_C^T D)^{-1}H(\Phi_C^T\hat{\boldsymbol{Q}})\right] \tag{47}$$

$$k = 1, \ldots, n_s$$

where the modal gradients $\partial\Phi_C/\partial v_{sk}$ were first encountered in Eq. (39). Differentiation of Eq. (44) with respect to control variables yields

$$\frac{\partial \boldsymbol{u}(0^+)}{\partial v_{ck}} = -(\Phi_C^T D)^{-1}\frac{\partial H}{\partial v_{ck}}(\Phi_C^T\hat{\boldsymbol{Q}}), \qquad k = 1, \ldots, \boldsymbol{n}_c \tag{48}$$

The modal rate gains along the diagonal H are themselves the control design variables, so that $\partial H_C/\partial v_{ck}$ is a matrix of all zeros, except for entry in the kth row and column which is equal to one. Actuator force derivatives at subsequent peak times can also be derived,[24] but were not considered in the examples to follow.

Example: Simply Supported Beam with Three Actuators

The performance index, Eq. (38), was minimized for the purpose of designing the IMSC modal gains for a uniform beam with three actuators, described in Ref. 3 (Fig. 1). The material properties are given in unspecified consistent units such that $EI/(mL^4) = 1$. The disturbance was a unit impulse at $0.43L$. The force actuators were located $0.15L$, $0.55L$, and $0.73L$ of the span. Coupled control to damp out the first eight modes produced a maximum actuator force of magnitude 3.8 units.[3] The first 12 modes were used to simulate the closed-loop response of the beam to the unit impulse. Time histories of the midspan deflection of the beam and the corresponding actuator forces appear in Fig. 2 for three cases using IMSC, described next.

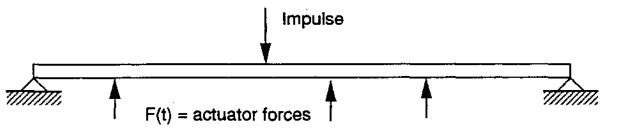

Fig. 1 Simply supported beam with three actuators.

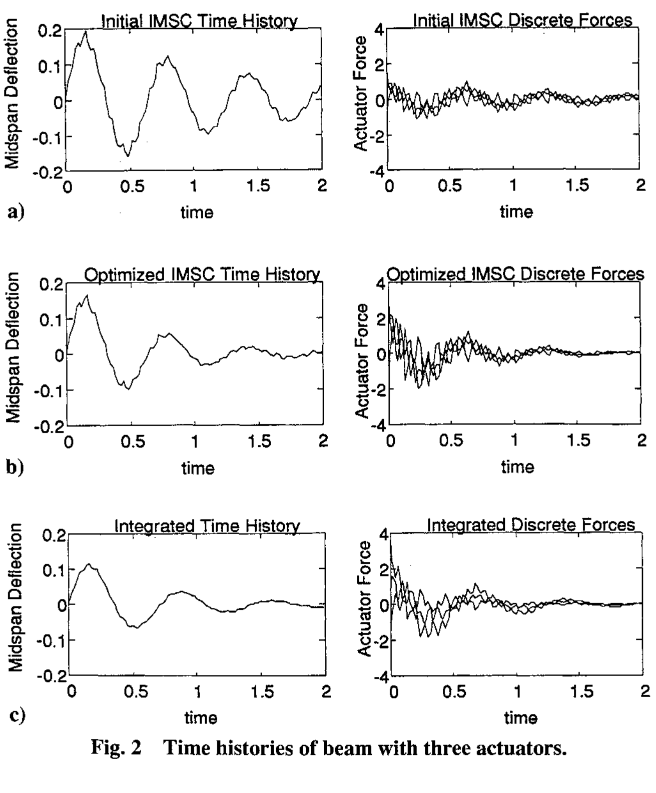

Fig. 2 Time histories of beam with three actuators.

Table 1 Independent modal space control gains for simply supported beam

Case	Mode i	ρ_i	g_i	h_i	ζ_i
Initial	1	1.0	0.4987	1.4133	0.0714
IMSC	2	1.0	0.4999	1.4142	0.0179
	3	1.0	0.5000	1.4142	0.0080
Optimized	1	0.1769	2.7874	3.3510	0.1674
IMSC	2	0.1391	3.5909	3.7910	0.0480
	3	0.1639	3.0500	3.4930	0.0197
Optimized	1	0.1955	2.517	3.186	0.1782
structure	2	0.0989	5.045	4.494	0.0614
and IMSC	3	0.1505	3.322	3.646	0.0199

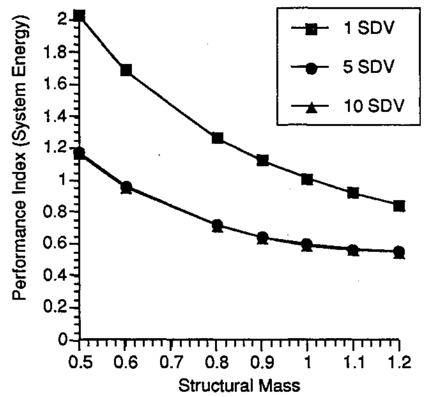

Fig. 3 Pareto optima for simply supported beam.

IMSC was used to control only the first three modes. For unit values of the modal control weighting factors r_r in Eq. (13), actuator forces were lower and their time history smoother than for coupled control, but at the expense of decreased damping. Next, IMSC modal gains were treated as design variables and Eq. (28) was minimized subject to constraints on each actuator force. The upper limit on the actuator forces, 3.8 units, was taken as the maximum value from the original example. Only a single call to an IMSL nonlinear optimization subroutine was needed. Results in Fig. 2b demonstrate higher damping and a 14% reduction in the maximum deflection without saturating the actuators. The modal control weighting factors and IMSC modal gains for each case are given in Table 1. This case represents the best performance possible by optimizing control gains alone without design structural variables. In other words, it is the optimal control for a fixed structure.

Next, structural design variables were introduced. To establish the relationship between bending and inertia properties (i.e., EI and mL), a particular cross-sectional shape was specified. A circular tube of outer radius $R = L/100$ and thickness $t = R/10$ was considered. Thickness was selected as the structural design variable while the outside radius remained fixed. The minimum thickness allowed was $t_{\min} = R/40$, whereas the maximum thickness was $t_{\max} = R/5$. To approximate the continuous distribution of material along the span, an Euler-Bernoulli beam was modeled using 100 finite elements. Each structural design variable controlled the thickness of a group of elements located symmetrically about the midspan. The performance index, Eq. (38), was minimized while requiring no increase in total mass and limiting the actuator forces. Also, the fundamental natural frequency was constrained to be at least 80% of its initial value. The integrated optimization problem may be stated as minimizing the system energy

$$\min_{v_c, v_s} J(\boldsymbol{v}_c, \boldsymbol{v}_s) \tag{49a}$$

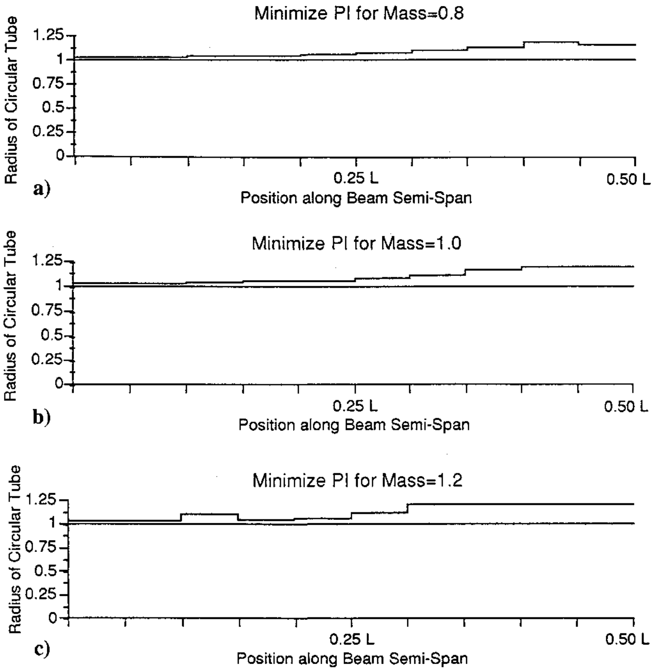

Fig. 4 Optimum beam profiles for 10 structural design variables.

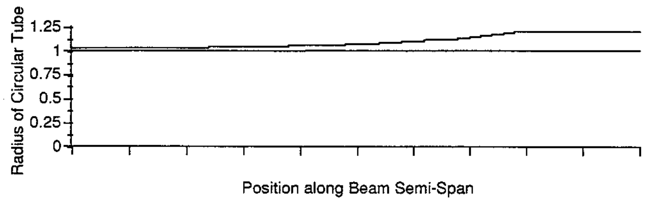

Fig. 5 Optimum beam profile for 50 structural design variables.

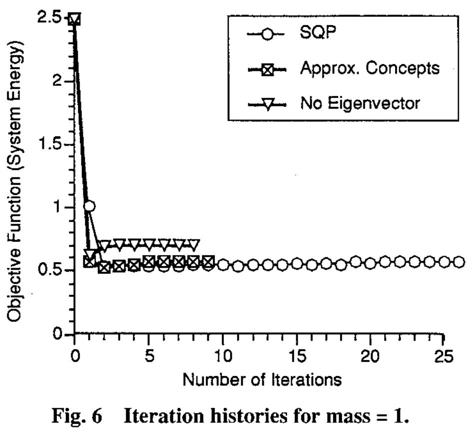

Fig. 6 Iteration histories for mass = 1.

subject to behavior constraints on nondimensional mass, fundamental frequency, and actuator constraints,

$$m(v_s) \leq 1 \tag{49b}$$

$$\omega_1^2(v_s) \geq 0.8\,(\omega_1^2)_0 \tag{49c}$$

$$u_j(v_c, v_s, t) \leq 3.8, \qquad j = 1, 2, 3 \tag{49d}$$

as well as side constraints

$$R/40 \leq v_{sk} \leq R/5, \qquad k = 1, 2, \ldots, 10 \tag{49e}$$

$$0 < v_{ck}, \qquad k = 1, 2, 3 \tag{49f}$$

Structural design variables v_s (tube thickness) and control design variables v_c (modal gains) were determined simultaneously. The time history in Fig. 2c indicates a 29% reduction in maximum amplitude without loss of damping or exceeding the actuator limit. This third case demonstrates the benefit of multiobjective optimization. Performance improvement over Fig. 2b was achieved by simultaneously determining control design variables and structural design variables.

Pareto optimal solutions were generated by parametrically varying the mass constraint. The resulting Pareto optimal sets in Fig. 3 show that not many structural design variables (SDV) were needed to approximate the Pareto optimal objective function values. Results for more than 10 structural design variables were not shown because they were indistinguishable from the curve for 10 variables. The same Pareto curves were generated by minimizing the mass subject to a constrained performance index. The resulting beam profiles in Fig. 4 for minimizing the performance index with 10 structural design variables were identical to those found by minimizing mass. Of course, the true optimum structural design entails a continuously varying cross section, simulated in Fig. 5 by using 50 structural design variables for mass constrained to unity (a refinement of Fig. 4b).

The effect of the eigenvector derivative terms appearing in Eq. (42) is demonstrated by the iteration histories given in Fig. 6. The curve labeled "SQP" represents a solution of the integrated optimization problem using sequential quadratic programming to solve Eqs. (49) directly. The curve labeled "Approx. Concepts" is a more efficient solution in which all structural and control functions are approximated by first- or second-order Taylor series.[24] The resulting approximate subproblem was solved iteratively until it converged to the exact solution found by SQP. The third curve, labeled "No Eigenvector," was an attempt to solve the integrated problem while ignoring the first term in the summation of Eq. (42). Without the eigenvector sensitivity, the solution converged to a distinctly different nonoptimal design with higher system energy. This alternate design was achieved only when using approximation concepts. The direct SQP algorithm failed to converge at all when eigenvector derivatives were ignored.

Conclusion

The integrated structural and control design problem was posed as a multiobjective optimization problem. The total energy integrated over time was identified as the performance index for constant feedback gain closed-loop control. Independent modal space control enabled a closed-form solution of the system energy as a function of open-loop frequencies, modal damping, and modal gains. Peak actuator forces were constrained to be within prescribed limits. Pareto optima for the multiobjective optimization problem were generated using the constraint method. The resulting scalar optimization problem was efficiently solved by forming an explicit approximate subproblem. Results revealed that eigenvector sensitivity was important to converging to the integrated optimum. More importantly, an integrated design approach clearly improved the performance of the closed-loop system without an increase in structural mass. Once the superiority of the integrated approach was established, Pareto optimal curves were developed to illustrate the extent to which structural mass changes could affect the closed-loop performance index.

Several important insights were gained from this research. First, a unique and physically meaningful performance index such as the total system energy lends itself to a fair comparison among candidate structural and control systems. Furthermore, IMSC offered the advantage not only of computational efficiency, but more importantly, it permitted the derivation of explicit expressions for the performance index and its derivatives. We infer from these derivatives that the total energy in the system imparted by an impulsive disturbance is reduced by softening the structure, i.e., by reducing its natural frequencies. Moreover, the eigenvector derivatives account for the effect structural changes have on the initial condition resulting from fixed external disturbances. The open-loop modes also determine the effectiveness of actuators. Thus, structural changes may help satisfy constraints on actuator forces.

Explicit actuator force constraints were incorporated as the physical means by which control gains were bounded. This approach of minimizing the system energy and structural mass subject to constraints on frequencies and actuator forces made integrated structural and control design a tractable problem. The use of IMSC and the selection of modal gains as the control design variables were the key to producing computational solutions for Pareto optimal designs, thus demonstrating the tradeoff between structural and control objectives.

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